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ABSTRACT

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The expansion of an infinitely long plasma column in a longitudinal magnetic field is considered. An initial equilibrium condition is postulated. The plasma is assumed to have finite conductivity. The analysis is based on the one-fluid hydrodynamic theory.

I. STATEMENT OF THE PROBLEM

The problem under study is the investigation of the motion of the surface of a plasma column which is assumed to be infinitely long. At an initial time $t = 0$, the plasma column has a finite radius R_0 inside which the plasma is postulated to have uniform temperature and pressure distributions. The initial kinetic pressure, denoted by p_0, balances the total external pressure: the sum of the kinetic pressure p_∞ and the magnetic pressure $B_0^2/2\mu_e$ (where B_0 is the applied longitudinal magnetic field and μ_e the permeability). Thus

$$p_0 = p_\infty + \frac{B_0^2}{2\mu_e} \tag{1}$$

The plasma is considered to have finite conductivity and hence it is expected that the magnetic field will begin to penetrate into the plasma column at $t = 0_+$. Soon after the diffusion process takes place, the initial equilibrium condition no longer exists and an expansion of the plasma column will be observed. To study such an expansion is the main interest of the following discussion.

The problem sounds much simpler than it is. In fact, the basic processes occurring in the present problem are so very involved as to preclude an exact calculation. Figure 1 presents a concise summary of the interacting factors involved in the analysis.

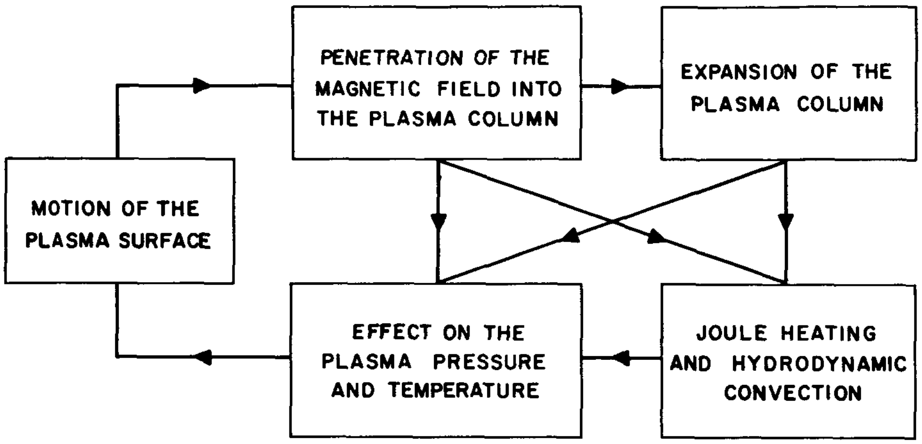

Fig. 1. Relationship of interacting factors

This diagram is self-explanatory and reveals one important point: the coupling between the moving boundary and the internal processes. To study the diffusion of the magnetic field, the Joule dissipation, the pressure variation, and many other phenomena taking place within the column, one needs information on the motion of the boundary of the plasma column. However, to determine the motion of such a boundary surface, one must first analyze the interior situation. This implies that we are going to deal with a free-boundary problem.

In the following investigation, we shall make a few basic assumptions:

1. The plasma density is sufficiently high throughout a significantly long time interval that the medium may be considered to be isotropic.

2. The plasma–gas interface is idealized as a "contact discontinuity." This is by no means true in a practical sense since near the interface a mixing layer always occurs because of the diffusion of the charged and neutral particles. However, if the "mixing layer" is stable and its thickness is small compared to the radius of the plasma column, the assumption is justifiable.

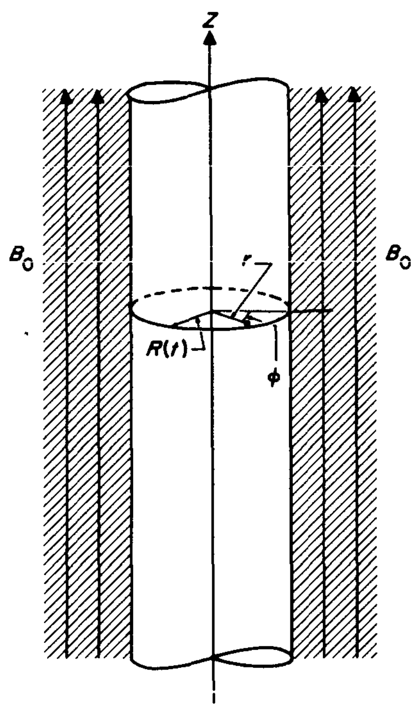

Fig. 2. Coordinates

3. The viscous friction and heat conduction phenomena are not important because of the uniformity of velocity and temperature distributions. The words "not important" must be understood only in comparison with the other quantities involved in equations regarding the conservation of momentum and energy.

4. The column is very long. If we introduce the cylindrical coordinate system as shown in Figure 2, all physical quantities will be considered independent of z.

II. DIFFUSION OF THE MAGNETIC FIELD

As mentioned previously, the most important part of the present problem is the coupling between the motion of the boundary and the internal processes. Of course, this coupling gives rise to a tremendous amount of mathematical complications, so that an exact solution is nearly out of the question. For this reason, some approximation must be made so that an analytical discussion of the problem will become possible.

Now, we shall first list all governing equations for the present problem. These equations may be grouped into two parts, i.e. the hydrodynamic equations and the electromagnetic field equations:

Hydrodynamic equations:

$$\frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} + \frac{\rho}{r} \frac{\partial r v_r}{\partial r} = 0 \qquad \text{(Continuity)} \tag{2}$$

$$\rho \left[\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} \right] = - \frac{\partial p}{\partial r} - \frac{\partial}{\partial r} \left(\frac{B_z^2}{2\mu_e} \right) \qquad \text{(Momentum)} \tag{3}$$

$$\rho \left[\frac{D\epsilon}{Dt} + p \frac{D}{Dt} \left(\frac{1}{\rho} \right) \right] = \frac{J_\phi^2}{\sigma} \qquad \text{(Energy)} \tag{4}$$

where $D/Dt = \partial/\partial t + v_r\, \partial/\partial r$ is the material derivative, ϵ is the specific internal energy of plasma, ρ is the plasma density, J_ϕ the azimuthal current, and σ and μ_e the electric conductivity and magnetic permeability.

Electromagnetic field equations:

$$\left.\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mu_e \mathbf{J} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \mathbf{J} &= \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \end{aligned}\right\} \tag{5}$$

where displacement current is ignored. Equation (5) may be combined in such a way that **J** and **E** are eliminated. Then it is possible to derive a single equation containing **B** and **v** only.

$$\nabla \times \nabla \times \mathbf{B} = -\nabla^2 \mathbf{B} = -\mu_e \sigma \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) \right] \tag{6}$$

In the present case, we have

$$\frac{\partial B_z}{\partial t} + v_r \frac{\partial B_z}{\partial r} + \frac{B_z}{r} \frac{\partial r v_r}{\partial r} = \frac{1}{\sigma \mu_e} \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial B_z}{\partial r} \tag{7}$$

Equations (2), (3), (4), and (7) should be solved simultaneously together with the following initial and boundary conditions:

At $t = 0$ $\quad v_r = 0$

$B_z = 0$ $\quad$ for $r < R(0)$

At $R = R(t)$ $\quad v_r = \dfrac{dR}{dt}$

$B_z = B_0$

At $r > R(t)$ $\quad p_t = p_\infty + \dfrac{B_0^2}{2\mu_e}$ $\quad$ (p_t is total pressure)

At $r = 0$ $\quad v_r = 0$

$\dfrac{\partial B_z}{\partial r} = 0$

where $R(t)$ denotes the surface of the plasma column. (For the time being the function $R(t)$ is unknown.)

Taking a closer look at these equations, it may be seen that one evident drawback of solving them is the coupling of the two groups of equations due to the hydromagnetic interaction. To remove this difficulty one may assume that the expansion of the plasma column is essentially a slow process. Thus, the hydrodynamic convection does not play an important role in the present problem. Since the motion is slow, for instance, we may ignore the inertia terms in the momentum equation. In other words, we assume that gradients of kinetic and magnetic pressures are the dominant terms. Similar approximations may be extended to the energy equation and the induction equation (Eq. 7). Then the magnetohydrodynamic coupling will disappear and considerable mathematical simplification may be achieved. In fact, this approximation has been used by many investigators working on the same subject, for example, Ref. 1.

However, in the present paper, we shall assume that the velocity distribution inside the plasma column follows the similarity form

$$v_r = \frac{r}{R(t)} \frac{dR}{dt} \tag{8}$$

which may be regarded as a reasonable approximation. This postulation has also been given by Braginsky (Ref. 1) and Artsimovich (Ref. 2) and many other authors. Hereafter, we will use Eq. (8) for solving Eq. (2), (3), (4), and (7), instead of ignoring those terms containing v_r completely. It is believed that this method shall provide a good approximation. Two reasons can be given:

First, Eq. (8) is a permissible expression physically. At least, for $r = 0$ and $r = R(t)$, it gives exact values of v_r. In the region $0 < r < R(t)$, (8) provides at least a good estimate of the order of magnitude. Intuitively, we believe v_r will not deviate significantly from what has been given by (8).

Second, we find that mathematically, in Eq. (2), (3), (4), and (7), those terms containing v_r are not expected to be very large. Therefore, the replacement of v_r by $(r/R)/(dR/dt)$, which represents at least the correct order of magnitude of v_r, should not alter the accuracy of the solution significantly.

Hence we shall proceed with expression (8) and solve Eq. (7) first. Before going further, we introduce the following transfomation of variables:

$$B_z(r, t) \rightarrow B(\eta, \tau) \tag{9}$$

where

$$\eta = \frac{r}{R(t)} \tag{10}$$

$$\tau = \int_0^t \frac{dt}{\sigma\mu_e R^2(t)} \tag{11}$$

It is simple in principle to transform Eq. (7) to the following form

$$\frac{\partial B'}{\partial t} = \frac{1}{\eta} \frac{\partial}{\partial \eta} \eta \frac{\partial B'}{\partial \eta} \tag{12}$$

where

$$B'(\eta, \tau) = B(\eta, \tau)\,\phi(\tau) \tag{13}$$

and

$$\phi(\tau) = R^2(t) \tag{14}$$

The corresponding boundary conditions are

$$\begin{array}{lll} \text{At } \tau = 0 & B' = 0 & \text{for } \eta < 1 \\ \text{At } \eta = 1 & B' = B_0\,\phi(\tau) & \\ \eta = 0 & \dfrac{\partial B'}{\partial \eta} = 0 & \end{array}$$

Therefore the mathematical problem reduces to the solution of an unsteady diffusion equation with time-dependent boundary condition $B_0\phi(\tau)$. No particular technique for solving this type of problem is necessary; the details are thus omitted here. The solution takes the form

$$B'(\eta, \tau) = -B_0 \sum_{n=1}^{\infty} \frac{2\beta_n J_0(\eta\beta_n)}{J_1(\beta_n)} e^{-\beta_n^2 \tau} \int_0^{\tau} e^{\beta_n^2 \lambda} \phi(\lambda)\, d\lambda \tag{15}$$

where β_n are the roots of $J_0(\beta) = 0$.

It is very simple to transform $B'(\eta, \tau)$ back to the (r, t) space. However, mathematically it is preferable to work in the (η, τ) space.

III. PRESSURE AND CONSERVATION OF MOMENTUM

We shall now discuss the pressure distribution within the plasma column. From Eq. (3) and (8), we obtain

$$\rho \frac{r}{R(t)} \frac{d^2R}{dt^2} = -\frac{\partial}{\partial r}\left[p + \frac{B_z^2}{2\mu_e}\right] \tag{16}$$

Hereafter, we shall assume

$$\rho \frac{d^2R}{dt^2} \simeq 0$$

and hence

$$p + \frac{B_z^2}{2\mu_e} = c \tag{17}$$

The integration constant c may be evaluated by the total pressure outside the plasma cylinder:

$$p + \frac{B_z^2}{2\mu_e} = p_\infty + \frac{B_0^2}{2\mu_e} \tag{18}$$

If $p^*(\eta, \tau)$ designates the pressure in the (η, τ) space, i.e. $p(r, t) = p^*(\eta, \tau)$, then from (15):

$$p^*(\eta,\tau) = p_\infty + \frac{B_0^2}{2\mu_e} - \frac{B_0^2}{2\mu_e \phi^2(\tau)} \left[\sum_{n=1}^{\infty} \frac{2\beta_n J_0(\beta_n \eta)}{J_1(\beta_n)} e^{-\beta_n^2 \tau} \int_0^\tau e^{\beta_n^2 \lambda} \phi(\lambda)\, d\lambda \right]^2$$

$$= p_0 - \frac{B_0^2}{2\mu_e \phi^2} \left[\sum_{n=1}^{\infty} \frac{2\beta_n J_0(\beta_n \eta)}{J_1(\beta_n)} e^{-\beta_n^2 \tau} \int_0^\tau e^{\beta_n^2 \lambda} (\lambda)\, d\lambda \right]^2 \tag{19}$$

One point that should be remarked is that numerically τ is of order $10^2 \sim 10^4$ when t is of the order of one second. $R(t)$ is about $1 \sim 10$ meters, and $\sigma \simeq 10^3$ mhos/m. Therefore, if we are interested in the behavior of $R(t)$ for $t \geq 1$ sec, it is justifiable to expand the integral involved in (19) as follows

$$e^{-\beta_n^2 \tau} \int_0^\tau e^{\beta_n^2 \lambda} \phi(\lambda)\, d\lambda \doteq \frac{1}{\beta_n^2} \left[\phi(\tau) - \frac{1}{\beta_n^2} \frac{d\phi}{d\tau} + \frac{1}{\beta_n^4} \frac{d^2\phi}{d\tau^2} + \cdots \right] \tag{20}$$

Again, since

$$\frac{d}{d\tau} = \sigma \mu_e R^2(t) \frac{d}{dt}$$

we may assume

$$\left(\frac{d\phi}{d\tau}\right)^2 = \frac{d^2\phi}{d\tau^2} = \frac{d^3\phi}{d\tau^3} + \cdots = 0$$

This makes it possible to reduce (19) to

$$p^*(\eta,\tau) = p_0 - \frac{B_0^2}{2\mu_e \phi^2}\left[\sum_{n=1}^{\infty} \frac{2J_0(\beta_n \eta)}{\beta_n J_1(\beta_n)}\left(\phi(\tau) - \frac{1}{\beta_n^2}\frac{d\phi}{d\tau}\right)\right]^2$$

$$= p_\infty + \frac{B_0^2}{2\mu_e}\left[\sum_{n=1}^{\infty} \frac{2J_0(\beta_n \eta)}{\beta_n^3 J_1(\beta_n)}\right]^2 \frac{2}{\phi(\tau)}\frac{d\phi}{d\tau} \tag{21}$$

where we have used the relation

$$\sum_{n=1}^{\infty} \frac{2J_0(\beta_n \eta)}{\beta_n J_1(\beta_n)} = 1$$

If we take the average value of $p^*(\eta,\tau)$ over the cross section, then (21) becomes

$$\bar{p}^*(\tau) = p_\infty + \frac{B_0^2}{2\mu_e}\frac{8}{\phi}\frac{d\phi}{d\tau}\, 2\int_0^1 \left[\sum_{n=1} \frac{J_0(\beta_n \eta)}{\beta_n^3 J_1(\beta_n)}\right]^2 \eta\, d\eta$$

$$= p_\infty + \frac{8B_0^2}{\mu_e}\frac{1}{\phi}\frac{d\phi}{d\tau}\sum_{n=1}\frac{1}{\beta_n^4} \tag{22}$$

IV. PRESSURE AND ENERGY BALANCE

The conservation of energy is described by the energy equation. In our analysis, the viscous dissipation and thermal conduction are ignored. This implies that the work done by pressure and heating due to Joule dissipation are far more important in the determination of the change of the internal energy of the plasma. The energy equation used in classical magnetohydrodynamic theory takes the form

$$\rho\left[\frac{D\epsilon}{Dt} + p\,\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = \mathbf{J}\cdot(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{23}$$

where ϵ is the specific internal energy and the other notations have been explained in section 2 above. Since

$$\rho\left[\frac{D\epsilon}{Dt} + p\,\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = \rho T\,\frac{DS}{Dt} = \rho T\left[\left(\frac{\partial S}{\partial p}\right)_\rho \frac{Dp}{Dt} - \left(\frac{\partial S}{\partial \rho}\right)_p \frac{D\rho}{Dt}\right] \tag{24}$$

where S is specific entropy.

It is possible to rewrite (23) as follows

$$\rho T\left[\left(\frac{\partial S}{\partial p}\right)_\rho \frac{Dp}{Dt} - \left(\frac{\partial S}{\partial \rho}\right)_p \rho\nabla\cdot\mathbf{v}\right] = \frac{J^2}{\sigma} = \frac{1}{\sigma\mu_e^2}(\nabla\times\mathbf{B})^2 \tag{25}$$

Sometimes, it is more convenient to express $(\partial S/\partial p)_\rho$ and $(\partial S/\partial\rho)_p$ in terms of other thermodynamic derivatives. For example,

$$\left(\frac{\partial S}{\partial p}\right)_\rho = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_p - \frac{c_p}{T}\,\frac{\left(\frac{\partial \rho}{\partial p}\right)_T}{\left(\frac{\partial \rho}{\partial T}\right)_p} \tag{26}$$

$$\left(\frac{\partial S}{\partial \rho}\right)_p = -\frac{1}{\rho^2}\left(\frac{\partial p}{\partial T}\right)_\rho - \frac{c_v}{T}\frac{\left(\frac{\partial p}{\partial \rho}\right)_T}{\left(\frac{\partial p}{\partial T}\right)_\rho} \tag{27}$$

If we assume that the plasma obeys approximately the equation of state[1] $p = \rho RT$, then

$$\left(\frac{\partial S}{\partial p}\right)_\rho = \frac{1}{T\rho(\gamma - 1)} \tag{28}$$

$$\left(\frac{\partial S}{\partial \rho}\right)_p = -\frac{p}{T\rho^2}\frac{\gamma}{(\gamma - 1)} \tag{29}$$

where

$$\gamma = \frac{c_p}{c_v}$$

Hence, in the present problem (25) takes the form

$$\frac{\partial p}{\partial t} + v_r\frac{\partial p}{\partial r} + \frac{\gamma p}{r}\frac{\partial r v_r}{\partial r} = (\gamma - 1)\frac{1}{\sigma\mu_e^2}\left(\frac{\partial B_z}{\partial r}\right)^2 \tag{30}$$

Again, we shall consider the transformation of p from the (r, t) space to (η, τ) space. With the previous assumption that

$$v_r = \eta\frac{dR}{dt}$$

[1] For more accurate calculation, one may use the Debye-Huckel theory or even more sophisticated considerations.

we obtain

$$\frac{\partial p'}{\partial \tau} + (\gamma - 1)\, p' \frac{d}{d\tau} \ln \phi(\tau) = \frac{(\gamma - 1)}{\mu_e \phi(\tau)} \left(\frac{\partial B'}{\partial \eta}\right)^2 \tag{31}$$

where

$$p' = p^* (\eta, \tau)\, \phi(\tau) = p\,(r,t)\, R^2\,(t)$$

$$\phi(\tau) = R^2\,(t)$$

$$B'(\eta, \tau) = B(\eta, \tau)\, \phi(\tau)$$

Equation (31) can be solved by considering $[(\gamma - 1)/\mu_e \phi(\tau)]\,(\partial B'/\partial\eta)^2$ as a forcing term. The solution is found to be

$$p'(\eta, \tau) = \phi^{(1-\gamma)}(\tau) \left[\int_0^{\tau} \frac{\gamma - 1}{\mu_e} \phi^{-(2-\gamma)}(\tau') \left(\frac{\partial B'}{\partial \eta}\right)^2 d\tau' + g(\eta)\right] \tag{32}$$

The function $g(\eta)$ may be determined by the initial condition

$$P'(\eta, 0) = P_0\, \phi\,(0) = \phi^{(1-\gamma)}\,(0)\, g(\eta)$$

$$\therefore\; g = p_0\, \phi^{\gamma}\,(0)$$

$$\therefore\; p^*(\eta, \tau) = \phi^{-\gamma}(\tau) \left[\int_0^{\tau} \frac{\gamma - 1}{\mu_e} \phi^{(\gamma-2)}(\tau') \left(\frac{\partial B'}{\partial \eta}\right)^2 d\tau' + p_0\, \phi^{\gamma}(0)\right] \tag{33}$$

In (33), the first term describes the pressure rise due to the phenomenon of ohmic heating, and the second term indicates the pressure drop corresponding to the expansion of the plasma cylinder.

From (15) the asymptotic expression at large time τ is

$$B'(\eta,\tau) = -B_0 \sum_{n=1}^{\infty} \frac{2J_0(\eta\beta_n)}{\beta_n J_1(\beta_n)} \left[\phi(\tau) - \frac{1}{\beta_n^2}\frac{d\phi}{d\tau} + \frac{1}{\beta_n^4}\frac{d^2\phi}{d\tau^2} + \cdots\right]$$

$$= -B_0\,\phi(\tau) + B_0 \sum_{n=1}^{\infty} \frac{2J_0(\eta\beta_n)}{\beta_n^3 J_1(\beta_n)} \frac{d\phi}{d\tau} + \cdots$$

Thus

$$\frac{\partial B'}{\partial \eta} = 2B_0 \sum_{n=1}^{\infty} \frac{J_1(\eta\beta_n)}{\beta_n^2 J_1(\beta_n)} \frac{d\phi}{d\tau} + \cdots$$

$$p^*(\eta,\tau) = \phi^{-\gamma}(\tau)\left[\frac{4(\gamma-1)B_0^2}{\mu_e} \int_0^{\tau} \phi^{(\gamma-2)}(\tau')\left(\sum_{n=1}^{\infty} \frac{J_1(\eta\beta_n)}{\beta_n^2 J_1(\beta_n)} \frac{d\phi}{d\tau} + \cdots\right)^2 d\tau' + \phi^{\gamma}(0)\,p_0\right]$$

If we keep only $d\phi/d\tau$, and drop $d^2\phi/d\tau^2$ etc., the averaged value of $p^*(\eta,\tau)$ over the cross section may be computed.

$$\overline{p}^*(\tau) = 2\int_0^1 p^*(\eta, \tau)\,\eta\, d\eta$$

$$= \frac{4(\gamma-1)B_0^2}{\mu_e}\,\phi^{-\gamma}(\tau)\sum_{n-1}^{\infty}\frac{1}{2\beta_n^4}\left[\left(1-\frac{1}{\beta_n^2}\right)J_1^2(\beta_n) + \beta_n^2\left(\frac{J_0(\beta_n)-J_2(\beta_n)}{2}\right)^2\right]$$

$$\times\ \frac{1}{J_1^2(\beta_n)}\int_0^{\tau}\phi^{(\gamma-2)}(\tau')\left(\frac{d\phi}{d\tau'}\right)^2 d\tau' + \phi^{-\gamma}(\tau)\,\phi^{\gamma}(0)\,p_0$$

$$= \frac{4(\gamma-1)B_0^2}{\mu_e}\,\phi^{-\gamma}(\tau)\sum_{n=1}^{\infty}\frac{1}{2\beta_n^4}\left[\left(1-\frac{1}{\beta_n^2}\right)+\frac{\beta_n^2}{4}\,\frac{J_2^2(\beta_n)}{J_1^2(\beta_n)}\right]$$

$$\times\ \int_0^{\tau}\left(\phi^{\gamma/2-1}\,\frac{d\phi}{d\tau'}\right)^2 d\tau' + \phi^{-\gamma}(\tau)\,\phi^{\gamma}(0)\,p_0$$

Since

$$\int_0^{\tau}\left(\frac{2}{\gamma}\right)^2\left(\frac{d\phi^{\gamma/2}}{d\tau'}\right)^2 d\tau' \simeq \left(\frac{\gamma^2}{4}\right)^{-1}\frac{d\phi^{\gamma/2}}{d\tau}\,\phi^{\gamma/2} = \left(\frac{\gamma^2}{4}\right)^{-1}\phi^{\gamma-1}\,\frac{d\phi}{d\tau}$$

we have

$$\overline{p}^*(\tau) = \frac{8(\gamma-1)B_0^2}{\gamma^2\mu_e}\sum_{n=1}^{\infty}\frac{1}{\beta_n^4}\left[\left(1-\frac{1}{\beta_n^2}\right)+\frac{\beta_n^2}{4}\,\frac{J_2^2(\beta_n)}{J_1^2(\beta_n)}\right]\frac{1}{\phi}\,\frac{d\phi}{d\tau} + \phi^{-\gamma}(\tau)\,\phi^{\gamma}(0)\,p_0 \qquad (34)$$

For simplicity, we shall denote

$$\frac{8(\gamma-1)B_0^2}{\gamma^2\mu_e}\sum_{n=1}^{\infty}\frac{1}{\beta_n^4}\left[\left(1-\frac{1}{\beta_n^2}\right)+\frac{\beta_n^2}{4}\frac{J_2^2(\beta_n)}{J_1^2(\beta_n)}\right]=\frac{8\Theta B_0^2}{\mu_e}$$

Hence

$$\overline{p}^*(\tau)=\frac{8\Theta B_0^2}{\mu_e\phi}\frac{d\phi}{d\tau}+\phi^{-\gamma}(\tau)\,\phi^{\gamma}(0)\,p_0 \tag{35}$$

V. MOTION OF THE INTERFACE

As mentioned in Section I, our essential interest is to study the motion of the surface of the plasma column. In the previous analysis, the function $R(t)$ (or $\phi(\tau)$) remained unknown. The purpose of this Section is to try to derive a final equation of $R(t)$ from which $R(t)$ may possibly be determined. The method of doing this is to establish first a fundamental relation from the previous results, based on the principle that the pressure determined from momentum conservation must be consistent with the solution obtained from the energy balance. In other words, expressions (22) and (35) should be consistent.

Now if we postulate that they are equal, then

$$p_\infty + \left(\frac{8B_0^2}{\mu_e}\sum_{n=1}^{\infty}\frac{1}{\beta_n^4}\right)\frac{1}{\phi}\frac{d\phi}{d\tau} = \frac{8\Theta B_0^2}{\phi\mu_e}\frac{d\phi}{d\tau} + \frac{p_0\phi^\gamma(0)}{\phi^\gamma(\tau)} \tag{36}$$

For simplicity we introduce the following notations

$$b = \frac{p_0\,\phi^\gamma(0)}{\dfrac{8B_0^2}{\mu_e}\left[\displaystyle\sum_{n=1}^{\infty}\frac{1}{\beta_n^4} - \Theta\right]}$$

$$c = \frac{p_\infty}{\dfrac{8B_0^2}{\mu_e}\left[\displaystyle\sum_{n=1}^{\infty}\frac{1}{\beta_n^4} - \Theta\right]}$$

Then (36) reduces to

$$\frac{d\phi}{\phi(b\,\phi^{-\gamma} - c)} = d\tau \tag{37}$$

Since $\sigma\mu_e \, \phi(\tau) = dt/d\tau$ we therefore can write

$$\int_{\phi(0)}^{\phi} \frac{\sigma\mu_e \, d\phi'}{b\phi'^{-\gamma} - c} = \int_0^t dt' = t \tag{38}$$

If in addition we introduce the following new variable ξ, i.e.

$$\frac{c}{b}\,\phi^{\gamma} = \frac{p_\infty}{p_0\,\phi^{\gamma}(0)}\,\phi^{\gamma}$$

and

$$M = \frac{8\sigma B_0^2}{p_\infty}\left(\frac{p_0}{p_\infty}\right)^{1/\gamma}\phi(0)\left(\sum_{n=1}^{\infty}\frac{1}{\beta_n^4} - \Theta\right) \tag{39}$$

the integral (38) will be transformed to the following form

$$t = M\int_{\alpha_0}^{\alpha}\frac{\xi'^{\gamma}\,d\xi'}{1-\xi'^{\gamma}} = M\left[\int_{\alpha_0}^{\alpha}\frac{d\xi'}{1-\xi'^{\gamma}} - (\alpha - \alpha_0)\right] \tag{40}$$

where

$$\alpha_0 = \left(\frac{p_\infty}{p_0}\right)^{1/\gamma} \qquad \alpha = \left(\frac{p_\infty}{p_0}\right)^{1/\gamma}\frac{\phi}{\phi(0)} = \left(\frac{p_\infty}{p_0}\right)^{1/\gamma}\frac{R^2(t)}{R^2(0)}$$

Since β_n are the roots of $J_0(\beta) = 0$, they can be listed as follows (Ref. 3):

$$\beta_1 = 2.4048256$$

$$\beta_2 = 5.5200781$$

$$\beta_3 = 8.6537279$$

$$\beta_4 = 11.7915339$$

$$\beta_5 = 14.9309177$$

$$\cdots\cdot$$

With these values, we obtain the following formula[2]

$$t = \frac{\sigma B_0^2}{4p_\infty}\left(\frac{p_0}{p_\infty}\right)^{1/\gamma} R^2(0)\left[\frac{(\gamma-1)^2+1}{\gamma^2}\right][f(\alpha) - f(\alpha_0)] \tag{41}$$

where

$$f(\alpha) = \int_0^\alpha \frac{d\xi}{1-\xi^\gamma} - \alpha$$

$$\alpha = \left(\frac{p_\infty}{p_0}\right)^{1/\gamma} \frac{R^2(t)}{R^2(0)}$$

$$\alpha_0 = \left(\frac{p_\infty}{p_0}\right)^{1/\gamma}$$

The function $f(\alpha)$ has been evaluated for three cases: $\gamma = 1.4$, $\gamma = 1.6$ and $\gamma = 1.8$. The results are given in Table 1 and Figure 3.

[2] The original accurate calculation gives

$$\sum \frac{1}{\beta_n^4} - \Theta = 0.0312268 - \frac{\gamma-1}{\gamma^2}\,0.0572453$$

For practical convenience, however, we have approximated

$$8\left[\sum \frac{1}{\beta_n^4} - \Theta\right] \simeq \frac{1}{4}\left[\frac{(\gamma-1)^2+1}{\gamma^2}\right]$$

Table 1. Values of $f(\alpha)$

α	$\gamma = 1.4$	$f(\alpha)$ $\gamma = 1.6$	$\gamma = 1.8$
0.05	0.00032	0.00016	0.00008
0.10	0.00170	0.00098	0.00057
0.15	0.00459	0.00286	0.00180
0.20	0.00938	0.00615	0.00408
0.25	0.01647	0.01123	0.00776
0.30	0.02631	0.01851	0.00776
0.35	0.03941	0.02845	0.02084
0.40	0.05641	0.04163	0.03119
0.45	0.07806	0.05872	0.04486
0.50	0.10535	0.08059	0.06264
0.55	0.13956	0.10837	0.08553
0.60	0.18239	0.14357	0.11489
0.65	0.23624	0.18829	0.15258
0.70	0.30461	0.24559	0.20135
0.75	0.39290	0.32021	0.26541
0.80	0.51015	0.42009	0.35180
0.85	0.76333	0.56004	0.48373
0.90	0.92048	0.77338	0.66080
0.95	1.37297	1.16631	1.00741

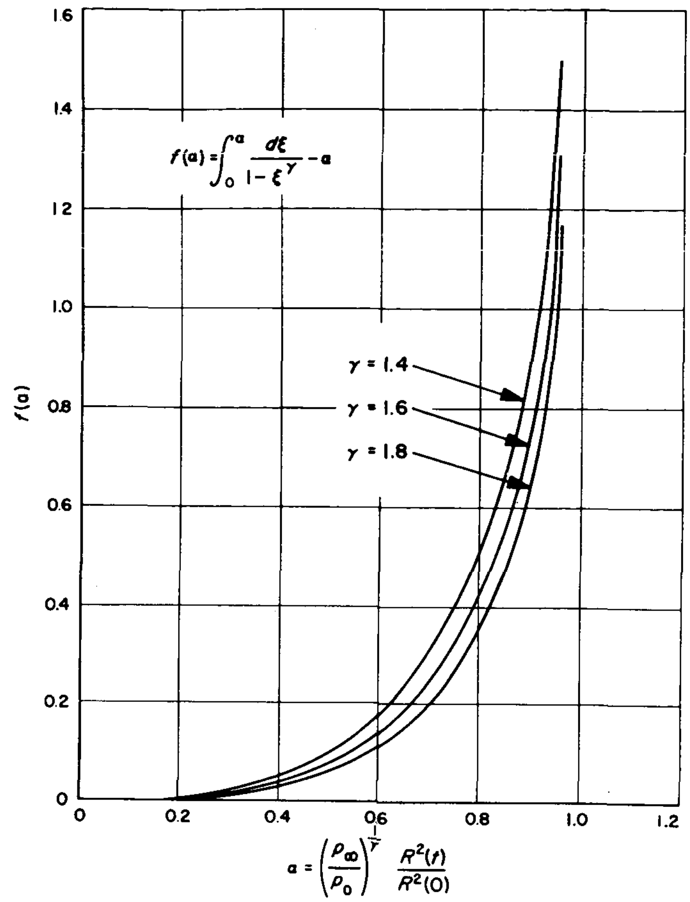

Fig. 3. Numerical results of the function $f(\alpha)$

In case $p_\infty \to 0$, Eq. (38) may be integrated immediately

$$t = \frac{8 B_0^2 \sigma}{p_0 \phi^{\gamma}(0)} \left[\sum_{n=1}^{\infty} \frac{1}{\beta_n^4} - \Theta \right] \frac{1}{\gamma + 1} \left[\phi^{\gamma+1} - \phi^{\gamma+1}(0) \right]$$

$$\simeq \frac{\sigma B_0^2}{p_0 R^{2\gamma}(0)} \frac{1}{4} \left[\frac{(\gamma-1)^2 + 1}{\gamma^2 (\gamma+1)} \right] \left[R^{2(\gamma+1)}(t) - R^{2(\gamma+1)}(0) \right] \tag{42}$$

VI. DISCUSSION AND CONCLUSIONS

In the previous discussions, we have kept only the first derivative $d\phi/d\tau$ and dropped all higher derivatives. This assumption is justified mainly by the fact that

$$\sigma \mu_e \phi \, d\tau = dt \tag{43}$$

where $\phi(\tau) = R^2(t)$ as defined previously.

It might appear that $d^2 R^2/dt^2$ is not negligible under the general situation. However, because of (42)

$$\frac{d^2\phi}{d\tau^2} = \sigma^2 \mu_e^2 \phi^2 \frac{d^2 R}{dt^2} \tag{44}$$

which is generally negligible. For example, if ϕ is of order 5 meters, then σ is about $10^3 \sim 10^4$ mhos/m, and $\mu_e = 4\pi \times 10^{-7}$, and we see that $\sigma^2 \mu_e^2 \phi^2$ is of order $10^{-3} \sim 10^{-5}$ numerically.

Again, during the expansion of the following quantity

$$e^{-\beta_n^2 \tau} \int_0^\tau e^{\beta_n^2 \lambda} \phi(\lambda) \, d\lambda$$

in Sections 3 and 4, we have used the "large τ" assumption, and dropped those terms like $\phi(0)/\beta_n^2 e^{\beta_n^2 \tau}$, $\phi(0)/\beta_n^4 e^{\beta_n^2 \tau}$, etc.

If we take a closer look at this approximation, it can be remarked that this assumption may be justified even when τ is of the order of unity, since $\beta_1^2 > 5$. In other words, this approximation may be justified for $t > 10^{-2}$ sec under the conditions illustrated in the past paragraph.

In the present discussion, we have treated the boundary of the plasma column as a contact discontinuity. Obviously this is not completely true. Because of the diffusion phenomena, we shall observe a mixing with finite width instead of a discontinuous interface. The previous discussion is not valid in this region because the plasma in this region cannot be described by the "one fluid" theory as the diffusion phenomena will be very important. Besides, since a large temperature gradient exists, thermal conduction cannot be ignored. If one wants to elaborate on the calculation of the fundamental characteristics of this mixing layer, more sophisticated theory should be used. So far we have assumed that what happens near the plasma-gas interface is not important to the determination of the kinetic pressure inside the plasma column. Again, we have also postulated that the amount of heat loss from the surface of the plasma column within a time interval δt

$$\delta t = \frac{\sigma B_0^2}{4p_\infty}\left(\frac{p_0}{p_\infty}\right)^{1/\gamma} R^2(0)\left[\frac{(\gamma-1)^2+1}{\gamma^2}\right]$$

has only a small effect on the total energy contained in the plasma. This assumption restricts us to the case that the value of σB_0^2 cannot be too large. If δt is large, the energy loss from the surface must be taken into account in the analysis.

With all these considerations, we conclude that the solutions of pressure from (17) and (33) in the vicinity of $r = R(t)$ should not be expected to be consistent. Nevertheless we consider that the expansion of such a plasma column depends mainly upon the average over-all pressure inside the plasma column and assume that the local pressure distribution near the surface $r = R$ is not important so far as the average value is concerned.

Again, in closing we remark that from (41) or (42) ($p_\infty \to 0$), a confinement time may be estimated. If one considers

$$f(\alpha) - f(\alpha_0) \approx 0.01$$

then

$$t_0 = \frac{\sigma B_0^2 R^2(0)}{4p_\infty}\left(\frac{p_0}{p_\infty}\right)^{1/\gamma}\left[\frac{(\gamma-1)^2+1}{\gamma^2}\right]^2 \frac{1}{100}$$

Furthermore, if we consider that $p_\infty \to 0$ and $R^{2\gamma}/R^{2\gamma}(0) \approx 0.01$

$$t_0' = \frac{\sigma B_0^2 R^{2(1-\gamma)}(0)}{4p_0}\left[\frac{(\gamma-1)^2+1}{\gamma^2(\gamma+1)}\right]\frac{1}{100}$$

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